Here are some important reminders about integrals from Calculus I and Calculus II. This sheet **is not** meant to be an introduction to integrals, and it **does not** pretend to cover everything necessary for this course. Beware that, while many integrals can be computed using these methods, this is (provably) not an exhaustive list of techniques. Additionally, many integrals can be computed only by using several of these techniques together.

Try practice problems to be sure that you are computationally competent. You can find many problems in the textbook, looking in the index for keywords to practice a specific technique.

Basic Properties

If functions f, g are continuous and a is constant, then the following hold.

1. $\displaystyle\int f(x) + g(x)\ dx = \int f(x)\ dx + \int g(x)\ dx$
2. $\displaystyle\int af(x)\ dx = a \int f(x)\ dx$

Basic Forms

Here are some basic indefinite integrals (i.e. antiderivatives).

3. $\displaystyle\int x^n\ dx = \frac{1}{n+1} x^{n+1} + C$ for all $n \neq -1$
4. $\displaystyle\int \cos(x)\ dx = \sin(x) + C$
5. $\displaystyle\int \sin(x)\ dx = -\cos(x) + C$
6. $\displaystyle\int \sec(x)\ dx = \ln|\sec(x) + \tan(x)| + C$
7. $\displaystyle\int \frac{1}{x}\ dx = \ln(x) + C$
8. $\displaystyle\int \exp(x)\ dx = \exp(x) + C$

Substitution Method

If f is an integrable function with antiderivative F and u is differentiable, then the following holds.

9. $\displaystyle\int f(u(x))u'(x)\ dx = F(u(x)) + C$

The hardest part of applying this method is recognising a good substitution; you can develop this skill with practice.

Many of the later techniques involve using this one too, so this one is absolutely fundamental.

By Parts Method

If u and v are differentiable functions, then the following holds.

10. $\displaystyle\int u(x)v'(x)\ dx = u(x)v(x) - \int v(x)u'(x)\ dx$

This is often written $\displaystyle\int u\ dv = uv - \int v\ du$. Again, practice will help develop the skill to recognise substitutions.

Trigonometric Substitutions

The following are useful guidelines for integrating products of trigonometric functions.

11. To compute $\displaystyle\int \cos^m(x)\sin^n(x)\ dx$ for m, n natural numbers, try the following.
    (a) If m is odd, try $u(x) = \sin(x)$ and use $\sin^2(x) + \cos^2(x) = 1$ to convert the remaining cosines.
    (b) If n is odd, try $u(x) = \cos(x)$ and use $\sin^2(x) + \cos^2(x) = 1$ to convert the remaining sines.
    (c) If m, n are both even, use $\cos^2(x) = \frac{1}{2}(1 + \cos(2x))$ and $\sin^2(x) = \frac{1}{2}(1 - \sin(2x))$, simplify, and reassess.
12. To compute $\displaystyle\int \sec^m(x)\tan^n(x)\ dx$ for m, n natural numbers, try the following.
    (a) If m is even, try $u(x) = \tan(x)$ and use $1 + \tan^2(x) = \sec^2(x)$ to convert the remaining secants.
    (b) If m, n are both odd, try $u(x) = \sec(x)$ and use $1 + \tan^2(x) = \sec^2(x)$ to convert the remaining tangents.
    (c) If m is odd but n is even, this integral is very hard; there is no simple-to-state method here.

Slight variations on the second trick can be used to compute $\displaystyle\int \csc^m(x)\cot^n(x)\ dx$ for m, n natural numbers.

Inverse Trigonometric Substitutions (a.k.a. Triangle Substitutions)

To compute integrals involving some complicated square roots, try the following methods. Draw triangles using the given trigonometric ratios to get the most out of this section.

13. If $\sqrt{a^2 - x^2}$ is involved in the integrand, try $\dfrac{x}{a} = \sin(\theta)$.
14. If $\sqrt{a^2 + x^2}$ is involved in the integrand, try $\dfrac{x}{a} = \tan(\theta)$.
15. If $\sqrt{x^2 - a^2}$ is involved in the integrand, try $\dfrac{x}{a} = \sec(\theta)$.

From the described substitution, you should be able to compute the appropriate differential in each case above. Using a triangle (determined by the substitution), you may be able to replace all of the nasty square roots with trigonometric ratios and your integral may reduce to a form computable from previous techniques.

Partial Fractions

16. To compute the integral of a rational function $f(x) = \dfrac{n(x)}{d(x)}$, use the following outline.
    (a) Use the Quotient–Remainder Theorem (a.k.a. "polynomial long division") to rewrite $f(x) = q(x) + \dfrac{r(x)}{d(x)}$ where either $r(x) = 0$ or $\deg(r) < \deg(d)$. Because $q(x)$ is a polynomial, we can compute its integral.
    (b) Factor $d(x) = p_1(x)^{m_1} p_2(x)^{m_2} \ldots p_k^{m_k}$ into powers of irreducible polynomials. Each p_i will be either a linear $ax + b$ or a quadratic $c(x - h)^2 + a^2$ with $a, c > 0$, and the p_i's are not multiples of one another.
    (c) Set up an equation based on your factorisation of $d(x)$ and these rules for each irreducible power $p(x)^m$.
        a. If $p(x) = ax + b$ is linear, add $\dfrac{A_1}{ax + b} + \dfrac{A_2}{(ax + b)^2} + \cdots + \dfrac{A_m}{(ax + b)^m}$.
        b. If $p(x) = (ax^2 + bx + c)^m$, add $\dfrac{B_1 x + C_1}{ax^2 + bx + c} + \dfrac{B_2 x + C_2}{(ax^2 + bx + c)^2} + \cdots + \dfrac{B_m x C_m}{(ax^2 + bx + c)^m}$.

        The resulting equation will have the form $\dfrac{r(x)}{d(x)} =$ [the sum of the stuff from above].
    (d) Multiply both sides by $d(x)$ and write the resulting polynomial in normal form (i.e. simplify it out).
    (e) Set coefficients equal and solve the resulting linear system (in the A_i, B_i, C_i from a previous step).
    (f) Integrate each resulting term using previous techniques.